

A NOTE ON TORSION BREUIL MODULES IN THE CASE

$$er = p - 1$$

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ABSTRACT. We prove that the category of unipotent torsion Breuil modules is an abelian category, under the condition $er = p - 1, r < p - 1$. As a side product, we point out that some of the approaches in [Car08] on torsion log-syntomic cohomologies cannot be generalized to $er = p - 1$.

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INTRODUCTION

Let p be a prime, k a perfect field of characteristic p , $W(k)$ the ring of Witt vectors, $K_0 = W(k)[\frac{1}{p}]$ the fraction field, K a finite totally ramified extension of K_0 , $e = e(K/K_0)$ the ramification index, \overline{K} a fixed algebraic closure, and $G_K = \text{Gal}(\overline{K}/K)$ the absolute Galois group. We will use $r \in \mathbb{Z}^+$ to denote an integer such that $er \leq p - 1$. Many of the results in this note are valid for $er \leq p - 1$, but we will be most interested in the case $er = p - 1$.

In this note, we will prove that the category $\underline{\mathcal{M}}^{r,u}$ (See Section 2 for the definition) of unipotent torsion Breuil modules is an abelian category when $er = p - 1$ and $r < p - 1$. We first show that the subcategory consisting of objects killed by p is abelian by proving that it is equivalent to another abelian category $\text{Mod}_{k[u]/u^p}^{\phi,u}$ (see Section 1). A dévissage argument will then finish the proof.

The original motivation for considering these categories was to see if we can generalize the work of [Car08] on torsion log-syntomic cohomologies to the case $er = p - 1$. However, during the work, we found out that the original approach of [Car08] cannot be generalized. We will point out the reason in Section 3.

As a final note, we hope that these results will be useful for studying (crystalline) Galois representations in the future.

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Notations: We use $[a_1, \dots, a_d]$ to denote a diagonal matrix with the elements in the bracket. We use A^T to denote the transpose of a matrix. In particular, $(e_1, \dots, e_d)^T$ is a column vector. We use a boldface letter to mean a column vector, e.g., \mathbf{e} or $\boldsymbol{\alpha}$. We use notations like $\oplus R\mathbf{e}$ to denote the space of R -span of vectors in \mathbf{e} , i.e., if $\mathbf{e} = (e_1, \dots, e_d)^T$, then $\oplus R\mathbf{e} = \oplus_{i=1}^d Re_i$. When the ring R is clear from the context, we can omit it and simply denote $\oplus \mathbf{e}$.

1. MODULES WITH FILTRATIONS

In this section, we define certain categories of modules with filtrations, and prove that they are abelian categories under some conditions.

Let π be a fixed uniformizer of K . Let $E(u) \in W(k)[u]$ be the minimal polynomial of π over K_0 , which is of degree e . Recall that S is the p -adic completion of the PD-envelope of $W(k)[u]$ with respect to the ideal $(E(u))$, which is a $W(k)[u]$ -subalgebra of $K_0[[u]]$, and $S = \{\sum_{i=0}^{\infty} a_i \frac{E(u)^i}{i!} \mid a_i \in W(k)[u], a_i \rightarrow 0 \text{ } p\text{-adically}\}$. S has a filtration $\{\text{Fil}^i S\}_{i \geq 0}$, where $\text{Fil}^i S$ is the p -adic completion of the ideal generated by all $\gamma_j(E(u)) = \frac{E(u)^j}{j!}$ with $j \geq i$. There is a Frobenius $\phi : S \rightarrow S$ which acts on $W(k)$ via Frobenius and sends u to u^p , and a $W(k)$ -linear differentiation N (called the monodromy operator) such that $N(u) = -u$. We denote $c = \frac{\phi(E(u))}{p}$, which is a unit in S . we also denote $S_n = S/p^n S$. Note that $\phi(\text{Fil}^i S) \subseteq p^i S$ for $1 \leq i \leq p-1$, and we denote $\phi_i := \frac{\phi}{p^i} : \text{Fil}^i S \rightarrow S$ for $1 \leq i \leq p-1$.

Let $S_1 = S/pS$, $\text{Fil}^r S_1 := \text{Fil}^r S + pS/pS \simeq \text{Fil}^r S/p\text{Fil}^r S$, and $\phi_r : \text{Fil}^r S_1 \rightarrow S_1$ the map induced from $\phi_r : \text{Fil}^r S \rightarrow S$.

Let $T_{ep} := S_1/\text{Fil}^p S_1 \simeq k[u]/u^{ep}$, $\text{Fil}^r T_{ep} := u^{er} \cdot k[u]/u^{ep}$, $\phi_r : \text{Fil}^r T_{ep} \rightarrow T_{ep}$ with $\phi_r(u^{er}) := (\frac{\phi(E(u)) - E(u)^p}{p})^r (\text{mod } p)$. Note that if $E(u) = u^e + pF(u)$, then $\frac{\phi(E(u)) - E(u)^p}{p} (\text{mod } p) = \overline{\phi(F(u))}$, which is a unit in T_{ep} because $F(0)$ is a unit in $W(k)$.

When $er \leq p-1$, $p \leq s \leq ep$, we define $T_s := k[u]/u^s$, and let $\text{Fil}^r T_s := u^{er} \cdot k[u]/u^s$, and $\phi_r : \text{Fil}^r T_s \rightarrow T_s$ such that $\phi_r(u^{er}) = (\overline{\phi(F(u))})^r$.

When $er \leq p-1$ and $p \leq s \leq ep$, it can be easily checked that the following diagram is commutative,

$$\begin{array}{ccccccc} \text{Fil}^r S & \longrightarrow & \text{Fil}^r S_1 & \longrightarrow & (u^{er})/u^{ep} & \longrightarrow & (u^{er})/u^s \\ \phi_r \downarrow & & \phi_r \downarrow & & \phi_r \downarrow & & \phi_r \downarrow \\ S & \longrightarrow & S_1 & \longrightarrow & T_{ep} = k[u]/u^{ep} & \longrightarrow & T_s = k[u]/u^s. \end{array}$$

In the following, we will just denote $\overline{\phi(F(u))}$ by c for simplicity.

Definition 1.0.1. For $er \leq p-1, p \leq s \leq ep$, the category $\text{Mod}_{k[u]/u^s}^\phi$ consists of objects $(\mathcal{M}, \text{Fil}^r \mathcal{M}, \phi_r)$ where

- \mathcal{M} is a finite free T_s -module.
- $\text{Fil}^r \mathcal{M}$ is a submodule of \mathcal{M} containing $u^{er} \mathcal{M}$.
- $\phi_r : \text{Fil}^r \mathcal{M} \rightarrow \mathcal{M}$ is a map such that $\phi_r(ax) = \phi(a)\phi_r(x)$ for any $a \in T_s, x \in \text{Fil}^r \mathcal{M}$, and the image generates \mathcal{M} .

Morphisms in the category are T_s -homomorphisms that are compatible with filtrations and ϕ_r .

A sequence $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_2 \rightarrow 0$ in $\text{Mod}_{k[u]/u^s}^\phi$ is called short exact if it is short exact as T_s -modules, and the sequence on filtrations $0 \rightarrow \text{Fil}^r \mathcal{M}_1 \rightarrow \text{Fil}^r \mathcal{M} \rightarrow \text{Fil}^r \mathcal{M}_2 \rightarrow 0$ is also short exact. In this case, we call \mathcal{M}_2 a quotient of \mathcal{M} .

The Cartier dual of \mathcal{M} is defined by $\mathcal{M}^\vee := \text{Hom}_{T_s}(\mathcal{M}, T_s)$,

$$\text{Fil}^r \mathcal{M}^\vee := \{f \in \mathcal{M}^\vee, f(\text{Fil}^r \mathcal{M}) \subseteq \text{Fil}^r T_s\},$$

and

$$\varphi_r^\vee : \text{Fil}^r \mathcal{M}^\vee \rightarrow \mathcal{M}^\vee, \text{ for all } x \in \text{Fil}^r \mathcal{M}, \varphi_r^\vee(f)(\varphi_r(x)) = \varphi_r(f(x)).$$

Note that $\varphi_r^\vee(f)$ is well defined since $\varphi_r(\text{Fil}^r \mathcal{M})$ generates \mathcal{M} .

- Lemma 1.0.2.** (1) *Given a $k[u]/u^s$ -module \mathcal{M} of finite type, suppose it can be written as a direct sum $\mathcal{M} = \oplus_{i=1}^d T_s m_i$ where $m_i \neq 0$ (\mathcal{M} is not necessarily free). Then for any submodule \mathcal{N} of \mathcal{M} , we can choose some nonzero elements $e_i \in \mathcal{M}, 1 \leq i \leq d$ such that $\mathcal{M} = \oplus_{i=1}^d T_s e_i$ and $\mathcal{N} = \oplus_{i=1}^d T_s u^{x_i} e_i$ where $0 \leq x_i < s$. Note that here we allow $u^{x_i} e_i$ to be 0.*
- (2) *Given a $k[u]/u^s$ -module \mathcal{M} of finite type, if $\mathcal{M} = \oplus_{i=1}^a \alpha_i = \oplus_{j=1}^b \beta_j$ where $\alpha_i, \beta_j \neq 0, \forall i, j$, then $a = b$.*

Proof. Statement (1) can be proved similarly as Lemma 3.2.1 of [Car06]. Statement (2) is easily deduced by an induction on $\min\{a, b\}$, using similar idea as the proof of (1). \square

Lemma 1.0.3. *The Cartier dual functor induces an anti-equivalence (thus a duality), and it transforms short exact sequences to short exact sequences.*

Proof. By Lemma 1.0.2, given any $\mathcal{M} \in \text{Mod}_{k[u]/u^s}^\phi$, there is a “base adaptée” for $\text{Fil}^r \mathcal{M}$. Then this proposition is similarly proved as Proposition V.3.1.6 of [Car05]. \square

For $\mathcal{M} \in \text{Mod}_{k[u]/u^s}^\phi$, let $(\phi^*)^1 \mathcal{M} = \phi^* \mathcal{M}$ be the T_s -span of $\phi_r(u^{er} \mathcal{M})$, and inductively, define $(\phi^*)^n \mathcal{M}$ as the T_s -span of $\phi_r(u^{er} (\phi^*)^{n-1} \mathcal{M})$. Let $\mathcal{M}^m := \cap_{n=1}^\infty (\phi^*)^n \mathcal{M}$, and $\text{Fil}^r \mathcal{M}^m := \mathcal{M}^m \cap \text{Fil}^r \mathcal{M}$.

Proposition 1.0.4. *\mathcal{M}^m is a finite free T_s -module, and $\text{Fil}^r \mathcal{M}^m = u^{er} \mathcal{M}^m$.*

Proof. \mathcal{M}^m is a submodule of \mathcal{M} , by Lemma 1.0.2, we can choose a basis e_1, \dots, e_d of \mathcal{M} , such that $\mathcal{M}^m = \oplus_{i=1}^a T_s u^{x_i} e_i$, where $a \leq d, 0 \leq x_i < s$. We claim that $x_i = 0$ for all i . To prove the claim, suppose otherwise, then without loss of generality, we can assume $x_1 > 0$ and is maximal among all x_i . By the definition of \mathcal{M}^m , $\phi_r(u^{er} \mathcal{M}^m)$ generates \mathcal{M}^m , it means that $(\phi_r(u^{er} u^{x_1} e_1), \dots, \phi_r(u^{er} u^{x_a} e_a))^T = C(u^{x_1} e_1, \dots, u^{x_a} e_a)^T$ for some invertible $a \times a$ -matrix C . Now

$$\begin{aligned} (\phi_r(u^{er} u^{x_1} e_1), \dots, \phi_r(u^{er} u^{x_a} e_a))^T &= (u^{px_1} \phi_r(u^{er} e_1), \dots, u^{px_a} \phi_r(u^{er} e_a))^T \\ &= [u^{px_1}, \dots, u^{px_a}](A, B)(e_1, \dots, e_a, \dots, e_d)^T, \end{aligned}$$

where A is an $a \times a$ -matrix and B is an $a \times (d - a)$ -matrix.

Thus, we have $[u^{px_1}, \dots, u^{px_a}]A = C[u^{x_1}, \dots, u^{x_a}]$. With this equality, we can easily show that all elements in the first row of C are divisible by u (using that x_1

is nonzero and maximal among all x_i), which contradicts that C is invertible! Thus our claim is proved, and \mathcal{M}^m is finite free.

Since we have already shown that \mathcal{M}^m is finite free, so we can choose a basis (e_1, \dots, e_a) of \mathcal{M}^m such that $\text{Fil}^r \mathcal{M}^m = \bigoplus_{i=1}^a T_s u^{x_i} e_i$ for some $0 \leq x_i \leq s$. Since clearly $\text{Fil}^r \mathcal{M}^m \supseteq u^{er} \mathcal{M}^m$, we must have $0 \leq x_i \leq er$. We claim that $x_i = er$ for all i . To prove the claim, suppose otherwise, and we can assume $x_1 < er$. Note that $u^{x_i} e_i \in \text{Fil}^r \mathcal{M}$, so $\phi_r(u^{x_i} e_i) \in \mathcal{M}$ (not necessarily in \mathcal{M}^m). Again we use the fact that $\phi_r(u^{er} \mathcal{M}^m)$ generates \mathcal{M}^m . So

$$\begin{aligned} (\phi_r(u^{er} e_1), \dots, \phi_r(u^{er} e_a))^T &= [u^{p(er-x_1)}, \dots, u^{p(er-x_a)}](\phi_r(u^{x_1} e_1), \dots, \phi_r(u^{x_a} e_a))^T \\ &= [u^{p(er-x_1)}, \dots, u^{p(er-x_a)}](A, B)(e_1, \dots, e_a, \dots, e_d)^T, \end{aligned}$$

where A is an $a \times a$ -matrix and B is an $a \times (d-a)$ -matrix. Thus $[u^{p(er-x_1)}, \dots, u^{p(er-x_a)}]A = C$ for some invertible $a \times a$ -matrix C . But this is impossible, because all elements on the first row of C will be divisible by u . So we have finished our proof. \square

Definition 1.0.5. $\mathcal{M} \in \text{Mod}_{k[u]/u^s}^\phi$ is called *multiplicative* if $\text{Fil}^r \mathcal{M} = u^{er} \mathcal{M}$, it is called *étale* if $\text{Fil}^r \mathcal{M} = \mathcal{M}$. It is called *nilpotent* if it has no nonzero multiplicative submodules, it is called *unipotent* if it has no nonzero étale quotients.

We will use $\text{Mod}_{k[u]/u^s}^{\phi, u}$ to denote the subcategory consisting of unipotent objects.

Proposition 1.0.6. For $\mathcal{M} \in \text{Mod}_{k[u]/u^s}^\phi$,

- (1) We have short exact sequences

$$0 \rightarrow \mathcal{M}^m \rightarrow \mathcal{M} \rightarrow \mathcal{M}^{\text{nil}} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{M}^{\text{uni}} \rightarrow \mathcal{M} \rightarrow \mathcal{M}^{\text{et}} \rightarrow 0.$$

where \mathcal{M}^m (resp. $\mathcal{M}^{\text{nil}}, \mathcal{M}^{\text{uni}}, \mathcal{M}^{\text{et}}$) is a multiplicative (resp. nilpotent, unipotent, étale) module. In fact, the second sequence is by taking Cartier dual of the first sequence, i.e., $\mathcal{M}^{\text{uni}} = (\mathcal{M}^{\vee, \text{nil}})^\vee$ and $\mathcal{M}^{\text{et}} = (\mathcal{M}^{\vee, m})^\vee$. Also, \mathcal{M}^m is the maximal multiplicative submodule of \mathcal{M} .

- (2) \mathcal{M} is multiplicative (resp. nilpotent) if and only if \mathcal{M}^\vee is étale (resp. unipotent), and vice versa.

Proof. It is similar to Theorem 2.3.7 of [Gao13], and much easier. Let \mathcal{M}^m be as in Proposition 1.0.4. Then as in the proof of *loc. cit.* shows, the quotient $\mathcal{M}^{\text{nil}} := \mathcal{M}/\mathcal{M}^m = \bigoplus_{j=a+1}^d e_j$ is finite free. Define $\text{Fil}^r \mathcal{M}^{\text{nil}} := \text{Fil}^r \mathcal{M}/\text{Fil}^r \mathcal{M}^m$ (which injects into $\mathcal{M}/\mathcal{M}^m$) with the induced ϕ_r . \mathcal{M}^{nil} is clearly a nilpotent module in the category $\text{Mod}_{k[u]/u^s}^\phi$. (2) is also easy to check. \square

Remark 1.0.7. Similarly as in Remark 2.3.8 of [Gao13], suppose $\text{Fil}^r \mathcal{M} = \bigoplus T_s \alpha$, and $\alpha = A \frac{\phi_r(\alpha)}{c^r}$, then \mathcal{M} is unipotent if and only if $\prod_{n=0}^\infty \phi^n(A) = 0$.

For $ep \geq t > s \geq p$, we can define a natural functor $\mathcal{M}_{t,s}$ from $\text{Mod}_{k[u]/u^t}^\phi$ to $\text{Mod}_{k[u]/u^s}^\phi$. For $\mathcal{M} \in \text{Mod}_{k[u]/u^t}^\phi$, let $\mathcal{M}_{t,s}(\mathcal{M}) := \mathcal{M}/u^s \mathcal{M}$, $\text{Fil}^r \mathcal{M}_{t,s}(\mathcal{M}) := \text{Fil}^r \mathcal{M}/u^s \mathcal{M}$ (which injects into $\mathcal{M}/u^s \mathcal{M}$), it is equipped with the induced ϕ_r . Note that ϕ_r is well defined since $\phi_r(u^s \mathcal{M}) = \phi(u^{s-er})\phi_r(u^{er} \mathcal{M}) = u^{p(s-er)}\phi_r(u^{er} \mathcal{M}) = 0$ because $p(s-er) \geq s$.

- Theorem 1.0.8.** (1) When $er = p - 1$ and $ep \geq t > s > p$, $\mathcal{M}_{t,s} : \text{Mod}_{k[u]/u^t}^\phi \rightarrow \text{Mod}_{k[u]/u^s}^\phi$ is an equivalence.
- (2) When $er = p - 1$ and $ep \geq t > p$, $\mathcal{M}_{t,p} : \text{Mod}_{k[u]/u^t}^{\phi,u} \rightarrow \text{Mod}_{k[u]/u^p}^{\phi,u}$ is an equivalence on the unipotent subcategories.
- (3) When $er \leq p - 1$ and $ep \geq t > s \geq p$, $\mathcal{M}_{t,s}$ sends short exact sequences to short exact sequences.
- (4) When $er < p - 1$ and $ep \geq t > s \geq p$, $\mathcal{M}_{t,s} : \text{Mod}_{k[u]/u^t}^\phi \rightarrow \text{Mod}_{k[u]/u^s}^\phi$ is an equivalence.

Before proving the theorem, we list several lemmas.

- Lemma 1.0.9.** (1) Given $\mathcal{M} \in \text{Mod}_{k[u]/u^s}^\phi$, we can choose a basis (e_1, \dots, e_d) of \mathcal{M} such that $\text{Fil}^r \mathcal{M} = \oplus T_s \alpha = \oplus_{i=1}^d \alpha_i = \oplus_{i=1}^d u^{x_i} e_i$. Let $\alpha = A(\frac{\phi_r(\alpha)}{c^r})$. Then there exists a matrix B such that $AB = BA = u^{er} \text{Id}$.
- (2) Given $\mathcal{M}_1, \mathcal{M}_2 \in \text{Mod}_{k[u]/u^s}^\phi$, let A_1, A_2 be the matrices constructed as in (1), then a morphism $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is determined by some matrix X such that $A_1 \phi(X) = X A_2$. X is not uniquely determined, but $\phi(X)$ is uniquely determined.

Proof. For (1), the existence of a matrix C (not unique) such that $CA = u^{er} \text{Id}$ is clear because $\text{Fil}^r \mathcal{M} \supseteq u^{er} \mathcal{M}$, but it does not guarantee $AC = u^{er} \text{Id}$ because u is not an integral element in $k[u]/u^s$. In our situation, $(u^{x_1} e_1, \dots, u^{x_d} e_d)^T = A(\frac{\phi_r(\alpha)}{c^r})$. So we have $[u^{x_1}, \dots, u^{x_d}](e_1, \dots, e_d)^T = A \frac{\phi_r(\alpha)}{c^r}$. So $A = [u^{x_1}, \dots, u^{x_d}] P$, where P is the invertible matrix such that $(e_1, \dots, e_d)^T = P \frac{\phi_r(\alpha)}{c^r}$. Thus we can take our $B = P^{-1} [u^{er-x_1}, \dots, u^{er-x_d}]$.

For (2), if we make $\text{Fil}^r \mathcal{M}_1 = \oplus T_s \alpha$, $\text{Fil}^r \mathcal{M}_2 = \oplus T_s \beta$ for some α and β , then we can choose a matrix X such that $f(\alpha) = X\beta$. \square

Lemma 1.0.10. Suppose $ep \geq t > s \geq p$. Let A_1, A_2, B_1, B_2, X, Q be matrices with coefficients in $k[u]/u^t$ such that $B_1 A_1 = u^{p-1} \text{Id}$ and $B_2 A_2 = u^{p-1} \text{Id}$, and we have a relation $A_1 \phi(X) - X A_2 = u^s Q$. Then there exists a matrix \hat{X} with coefficients in $k[u]/u^t$ such that $A_1 \phi(\hat{X}) = \hat{X} A_2$ and $\phi(\hat{X}) \equiv \phi(X) \pmod{u^s}$, if either of the following conditions is satisfied:

- (1) $s > p$.
- (2) $s = p$ and $\prod_{i=0}^\infty \phi^i(A_1) = 0$.

Proof. Let

$$\hat{X} = X + u(u^{s-p} Q B_2 + \sum_{n=0}^\infty (\prod_{i=0}^n \phi^i(A_1) \phi^{n+1}(u^{s-p} Q) \prod_{j=n+1}^0 \phi^j(B_2))).$$

Here $\prod_{j=n+1}^0 \phi^j(B_2)$ means $\phi^{n+1}(B_2) \phi^n(B_2) \cdots \phi(B_2) B_2$. \square

Lemma 1.0.11. Suppose $ep \geq t > s \geq p$. Let A_1, A_2, B_1, B_2, X be matrices with coefficients in $k[u]/u^t$ such that $B_1 A_1 = B_2 A_2 = u^{p-1} \text{Id}$, and we have a relation $A_1 \phi(X) = X A_2$. Suppose $\phi(X) \equiv 0 \pmod{u^s}$, then $\phi(X) = 0$ if either of the following conditions is satisfied:

- (1) $s > p$.
- (2) $s = p$ and $\prod_{i=0}^\infty \phi^i(A_1) = 0$.

Proof. For the second condition, since $\phi(X) \equiv 0 \pmod{u^p}$, we can let $X = uY$, then $A_1 u^p \phi(Y) = uY A_2$. Multiply B_2 on both sides, then we have $u^p(A_1 \phi(Y) B_2 - Y) = 0$, so $A_1 \phi(Y) B_2 - Y = u^{t-p} Q$ for some matrix Q . We claim that the matrix equation $A_1 \phi(Y) B_2 - Y = u^{t-p} Q$ with indeterminate Y has a unique solution. Suppose we have two solutions Y_1, Y_2 , and let $Z = Y_1 - Y_2$, then $Z = A_1 \phi(Z) B_2$, so $Z = A_1 \phi(Z) B_2 = A_1 \phi(A_1) \phi^2(Z) \phi(B_2) B_2 = \dots = 0$ because $\prod_{i=0}^{\infty} \phi^i(A_1) = 0$. And the unique solution is

$$Y = u^{t-p} Q + \sum_{n=0}^{\infty} \left(\prod_{i=0}^n \phi^i(A_1) \phi^{n+1}(u^{t-p} Q) \prod_{j=n}^0 \phi^j(B_2) \right).$$

So we have $Y = u^{t-p} W$. Now $X = u^{t-p+1} W$, so $u^{p(t-p+1)}$ divides $\phi(X)$. Since $p(t-p+1) \geq t$, so u^t divides $\phi(X)$.

For the first condition, we can set $X = u^{\lceil \frac{s}{p} \rceil} Y$, and then it follows from a similar argument as above. \square

Proof. (Proof of Theorem 1.0.8) We first prove essential surjectivity of both statement (1) and (2). Given $\mathcal{M}_s \in \text{Mod}_{k[u]/u^s}^\phi$ for $s \geq p$, suppose $\text{Fil}^r \mathcal{M}_s = \oplus_{i=1}^d T_s \alpha_i$, and $(\alpha_1, \dots, \alpha_d)^T = A(\phi_r(\alpha_1), \dots, \phi_r(\alpha_d))^T = A(e_1, \dots, e_d)^T$, then there exists B such that $BA = u^{p-1} \text{Id}$. Take any lift \hat{A}, \hat{B} of A, B respectively with elements in $k[u]/u^t$, then $\hat{B}\hat{A} = u^{p-1} \text{Id} + u^s Q$ for some Q , so $[(\text{Id} + u^{s-p+1} Q)^{-1} \hat{B}](\hat{A}) = u^{p-1}$. Now define $\mathcal{M}_t = \oplus_{i=1}^d T_t \hat{e}_i$, $\text{Fil}^r \mathcal{M}_t = \sum_{i=1}^d T_t(\hat{\alpha}_i)$ with $(\hat{\alpha}_1, \dots, \hat{\alpha}_d)^T = \hat{A}(\hat{e}_1, \dots, \hat{e}_d)^T$, and $\phi_r(\hat{\alpha}_i) = \hat{e}_i$. Then clearly \mathcal{M}_t is a preimage of \mathcal{M}_s .

Now we prove the full faithfulness of both statement (1) and (2). Let $\mathcal{M}_{t_1}, \mathcal{M}_{t_2} \in \text{Mod}_{k[u]/u^t}^\phi$, and $\mathcal{M}_{s_1} = \mathcal{M}_{t,s}(\mathcal{M}_{t_1}), \mathcal{M}_{s_2} = \mathcal{M}_{t,s}(\mathcal{M}_{t_2})$. We need to prove that $h : \text{Hom}(\mathcal{M}_{t_1}, \mathcal{M}_{t_2}) \rightarrow \text{Hom}(\mathcal{M}_{s_1}, \mathcal{M}_{s_2})$ is bijective. Let A_1, A_2 be the matrix for $\mathcal{M}_{t_1}, \mathcal{M}_{t_2}$ respectively as in Lemma 1.0.9, then $A_{1,s} \equiv A_1 \pmod{u^s}, A_{2,s} \equiv A_2 \pmod{u^s}$ are the matrix for $\mathcal{M}_{s_1}, \mathcal{M}_{s_2}$. To show surjectivity of h , given any morphism in $\text{Hom}(\mathcal{M}_{s_1}, \mathcal{M}_{s_2})$ is equivalent to a matrix $X_s \in \text{Mat}(k[u]/u^s)$ such that $A_{1,s} \phi(X_s) = X_s A_{2,s}$, now lift X_s to any $X \in \text{Mat}(k[u]/u^t)$, so $A_1 \phi(X) - X A_2 = u^s Q$, and we can apply Lemma 1.0.10 to conclude surjectivity. To show injectivity of h , suppose a morphism in $\text{Hom}(\mathcal{M}_{t_1}, \mathcal{M}_{t_2})$ maps to 0. This morphism is determined by some X as in the conditions of Lemma 1.0.11, so the morphism is itself 0.

For (3), given a short exact sequence $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_2 \rightarrow 0$ in $\text{Mod}_{k[u]/u^t}^\phi$, since these are finite free modules, it is clear that $0 \rightarrow \mathcal{M}_1/u^s \mathcal{M}_1 \rightarrow \mathcal{M}/u^s \mathcal{M} \rightarrow \mathcal{M}_2/u^s \mathcal{M}_2 \rightarrow 0$ is still short exact. For the filtration sequence, $\text{Fil}^r \mathcal{M}_1/u^s \mathcal{M}_1 \rightarrow \text{Fil}^r \mathcal{M}/u^s \mathcal{M} \rightarrow \text{Fil}^r \mathcal{M}_2/u^s \mathcal{M}_2$, the second map is clear surjective. Since $u^s \mathcal{M}_1 \subseteq u^s \mathcal{M} \cap \text{Fil}^r \mathcal{M}_1 \subseteq u^s \mathcal{M} \cap \mathcal{M}_1 = u^s \mathcal{M}_1$, so $u^s \mathcal{M} \cap \text{Fil}^r \mathcal{M}_1 = u^s \mathcal{M}_1$, and the first map is injective. Exactness in the center is also easily checked.

For (4), it can be similarly proved as (1) and (2) by modifying Lemma 1.0.10 and Lemma 1.0.11 (which in fact becomes easier). \square

Lemma 1.0.12. *Given a short exact sequence $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_2 \rightarrow 0$ in $\text{Mod}_{k[u]/u^s}^\phi$, then \mathcal{M} is unipotent if and only if both \mathcal{M}_1 and \mathcal{M}_2 are unipotent.*

Proof. Let A_1, A_2 be the matrix for $\mathcal{M}_1, \mathcal{M}_2$ respectively as in Lemma 1.0.9. Then one can find some basis for \mathcal{M} with the corresponding matrix $A = \begin{pmatrix} A_1 & 0 \\ * & A_2 \end{pmatrix}$. Then apply Remark 1.0.7. \square

Theorem 1.0.13. *When $er \leq p - 1$,*

- (1) $\text{Mod}_{k[u]/u^p}^\phi$ *is an abelian category.*
- (2) $\text{Mod}_{k[u]/u^p}^{\phi, u}$ *is an abelian subcategory.*

Proof. For (1), when $er < p - 1$, just as pointed out in Theorem 3.5.1 of [Car06], the proof is verbatim as that of Corollary 2.2.3.2 in [Bre98]. In fact, it also works for $er = p - 1$. But here, we give a more direct proof of this fact for all $er \leq p - 1$ (without using the category C_k in [Bre98]).

First, we show that if $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a morphism in $\text{Mod}_{k[u]/u^p}^\phi$, then $f(\mathcal{M}_1)$ is finite free. We use notations from Lemma 1.0.9, then $f(\mathcal{M}_1)$ is generated by $\phi(X)(f_1, \dots, f_n)$, where (f_1, \dots, f_n) is a basis of \mathcal{M}_2 , and X is a matrix with coefficient in $k[u]/u^p$. Since $\phi(u) = u^p = 0$ in $k[u]/u^p$, so $\phi(X)$ is in fact a matrix with coefficient in k . Thus we can easily show that $f(\mathcal{M}_1)$ is finite free, and so $(f(\mathcal{M}_1), f(\text{Fil}^r \mathcal{M}_1), \phi_r)$ is an object in $\text{Mod}_{k[u]/u^p}^\phi$.

Now, we show that if $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a morphism in $\text{Mod}_{k[u]/u^p}^\phi$, then $f(\text{Fil}^r \mathcal{M}_1) = f(\mathcal{M}_1) \cap \text{Fil}^r \mathcal{M}_2$. Since we have shown that $f(\mathcal{M}_1) \in \text{Mod}_{k[u]/u^p}^\phi$, we can and do assume that $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is injective. By Lemma 1.0.2, we can assume that $\text{Fil}^r \mathcal{M}_2$ is a direct sum of the form $\oplus_{i=1}^n T_p \alpha_i$, and $\text{Fil}^r \mathcal{M}_1 = \oplus_{i=1}^n T_p u^{x_i} \alpha_i$. Suppose $x_i = 0$ for $0 \leq i \leq a$, and $x_i > 0$ for $a + 1 \leq i \leq n$. Since $\phi_r(u^{x_i} \alpha_i) = \phi(u^{x_i}) \phi_r(\alpha_i) = 0$ for $x_i > 0$, so \mathcal{M}_1 is generated by $\phi_r(\alpha_i)$ for $0 \leq i \leq a$. Thus \mathcal{M}_1 is of rank a , and we can choose a basis (e_1, \dots, e_a) of \mathcal{M}_1 , such that $\text{Fil}^r \mathcal{M}_1 = \oplus_{i=1}^a u^{y_i} T_p e_i = \oplus_{i=1}^n T_p u^{x_i} \alpha_i$. By Lemma 1.0.2(2), we must have $\text{Fil}^r \mathcal{M}_1 = \oplus_{i=1}^a T_p \alpha_i$. Then we can easily deduce that $f(\text{Fil}^r \mathcal{M}_1) = f(\mathcal{M}_1) \cap \text{Fil}^r \mathcal{M}_2$.

Now, we show that if $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a morphism in $\text{Mod}_{k[u]/u^p}^\phi$, then the kernel \mathcal{K} with $\text{Fil}^r \mathcal{K} := \mathcal{K} \cap \text{Fil}^r \mathcal{M}_1$ and naturally induced ϕ_r is an object in $\text{Mod}_{k[u]/u^p}^\phi$. Since we have shown that $f(\mathcal{M}_1) \in \text{Mod}_{k[u]/u^p}^\phi$, we can and do assume that $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is surjective and $f(\text{Fil}^r \mathcal{M}_1) = \text{Fil}^r \mathcal{M}_2$. Suppose that the rank of \mathcal{M}_1 and \mathcal{M}_2 is d and n respectively. By Lemma 1.0.2(1), we can take (e_1, \dots, e_d) a basis of \mathcal{M}_1 , such that $\mathcal{K} = \oplus_{i=1}^d T_p u^{x_i} e_i$ for some $0 \leq x_1 \leq \dots \leq x_d \leq p$. Then $\mathcal{M}_2 = \oplus_{i=1}^d T_p \bar{e}_i$ where \bar{e}_i is u^{x_i} -torsion. Since \mathcal{M}_2 is finite free, we conclude that $x_1 = \dots = x_{d-n} = 0$ and $x_{d-n+1} = \dots = x_d = p$. Thus \mathcal{K} is finite free over $k[u]/u^p$ of rank $d - n$. Now, clearly $\text{Fil}^r \mathcal{K} \supseteq u^{er} \mathcal{K}$, and since ϕ_r and f commute, $\phi_r(\text{Fil}^r \mathcal{K}) \subseteq \mathcal{K}$. By Lemma 1.0.2, we can suppose $\text{Fil}^r \mathcal{K} = \oplus_{i=1}^{d-n} \alpha_i$, $\text{Fil}^r \mathcal{M}_2 = \oplus_{j=1}^n \beta_j$. For any $1 \leq j \leq n$, take $\hat{\beta}_j \in \text{Fil}^r \mathcal{M}_1$ such that $f(\hat{\beta}_j) = \beta_j$. Then $(\alpha_i, \hat{\beta}_j)_{i,j}$ generate $\text{Fil}^r \mathcal{M}_1$, and so $(\phi_r(\alpha_i), \phi_r(\hat{\beta}_j))_{i,j}$ generate \mathcal{M}_1 . Thus $(\phi_r(\alpha_i))_i$ generate \mathcal{K} .

Now we show that if $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a morphism in $\text{Mod}_{k[u]/u^p}^\phi$, then the cokernel \mathcal{N} with naturally induced Fil^r and ϕ_r is an object in $\text{Mod}_{k[u]/u^p}^\phi$. Again, we can and do assume that f is an injective morphism. Then by Lemma 1.0.2, we can suppose $\text{Fil}^r \mathcal{M}_2 = \oplus \alpha_i$, and $\text{Fil}^r \mathcal{M}_1 = \oplus u^{x_i} \alpha_i$ for $x_i = 0$ when $0 \leq i \leq a$, and $x_i > 0$ when $i > a$. Since $\phi_r(u^{x_i} \alpha_i) = 0$ if $x_i > 0$, similarly as the end of the

second paragraph of the proof shows, we must have $\text{Fil}^r \mathcal{M}_1 = \oplus_{i=1}^a \alpha_i$. Then it is easy to deduce that $\text{Fil}^r \mathcal{N} = \oplus_{i=a+1}^d \alpha_i$, \mathcal{N} is finite free and is in fact generated by $\phi_r(\alpha_i)$, $a+1 \leq i \leq d$.

For (2), use (1) and Lemma 1.0.12. \square

Corollary 1.0.14. (1) When $er < p-1$, $\text{Mod}_{k[u]/u^s}^\phi$ is an abelian category for $ep \geq s > p$.

(2) When $er = p-1$, $\text{Mod}_{k[u]/u^s}^{\phi, u}$ is an abelian category for $ep \geq s > p$.

Proof. Combine Theorem 1.0.8 and Theorem 1.0.13. Note that we have given a new proof to Corollary 3.5.7 of [Car06]. \square

2. TORSION BREUIL MODULES

In this section, we prove that the category of unipotent torsion Breuil modules is an abelian category when $er = p-1$, $r < p-1$.

Let $\underline{\mathcal{M}}^r$ be the category consisting of objects $(\mathcal{M}, \text{Fil}^r \mathcal{M}, \phi_r, N)$ (called torsion Breuil modules) where

- (1) $\mathcal{M} = \oplus_{i \in I} S_{n_i}$ for a finite set I .
- (2) $\text{Fil}^r \mathcal{M}$ is an S -submodule which contains $\text{Fil}^r S \cdot \mathcal{M}$.
- (3) $\phi_r : \text{Fil}^r \mathcal{M} \rightarrow \mathcal{M}$ is a Frobenius-semi-linear map such that $\phi_r(sx) = c^{-r} \varphi_r(s) \varphi_r(E(u)^r x)$ for $s \in \text{Fil}^r S$ and $x \in \mathcal{M}$, and the image of ϕ_r generates \mathcal{M} .
- (4) $N : \mathcal{M} \rightarrow \mathcal{M}$ is a $W(k)$ -linear map such that
 - $N(sx) = N(s)x + sN(x)$ for all $s \in S, x \in \mathcal{M}$.
 - $E(u)N(\text{Fil}^r \mathcal{M}) \subset \text{Fil}^r \mathcal{M}$.
 - The following diagram is commutative:

$$\begin{array}{ccc} \text{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M} \\ E(u)N \downarrow & & \downarrow cN \\ \text{Fil}^r \mathcal{M} & \xrightarrow{\phi_r} & \mathcal{M}. \end{array}$$

Morphisms in the category are S -linear maps that are compatible with Fil^r , ϕ_r and N .

Let $\underline{\mathcal{M}}^{r, \phi}$ be the category similar to $\underline{\mathcal{M}}^r$ but without N , i.e., $\underline{\mathcal{M}}^{r, \phi}$ consists of objects $(\mathcal{M}, \text{Fil}^r \mathcal{M}, \phi_r)$ satisfying (1), (2) and (3) above.

We denote the subcategory of $\underline{\mathcal{M}}^r$ consisting of objects killed by p by $\text{Mod}_{S_1}^{\phi, N}$, and the subcategory of $\underline{\mathcal{M}}^{r, \phi}$ consisting of objects killed by p by $\text{Mod}_{S_1}^\phi$.

When $r < p-1$, by the isomorphism $S_1/\text{Fil}^p S_1 \simeq k[u]/u^{ep}$, there is a natural functor $\text{Mod}_{S_1}^\phi \rightarrow \text{Mod}_{k[u]/u^{ep}}^\phi$ by sending $(\mathcal{M}, \text{Fil}^r \mathcal{M}, \phi_r)$ to $(\mathcal{M}/\text{Fil}^p S_1 \mathcal{M}, \text{Fil}^r \mathcal{M}/\text{Fil}^p S_1 \mathcal{M}, \phi_r)$. Note that this functor can be defined only when $r < p-1$, because we need to have $\phi_r(\text{Fil}^p S_1 \cdot \mathcal{M}) = 0$.

Recall that in Definition 2.5.3 of [Gao13], for $\mathcal{M} \in \text{Mod}_{S_1}^\phi$, if $\text{Fil}^r \mathcal{M} = \oplus \alpha + \text{Fil}^p S_1 \mathcal{M}$, $\frac{\phi_r(\alpha)}{c^r} = e$, $\alpha = Ae$ with $A \in \text{Mat}(S_1)$. Then \mathcal{M} is called *unipotent* (with respect to α and e) if $\prod_{n=1}^\infty \phi^n(A) = 0$. The definition in [Gao13] is only stated for $r = p-1$, but in fact it works for any $r \leq p-1$ (and any e), and it can be easily checked that the definition of unipotency is independent of choice of α and e . Denote the unipotent subcategory by $\text{Mod}_{S_1}^{\phi, u}$. An object $\mathcal{M} \in \text{Mod}_{S_1}^{\phi, N}$ is called

unipotent if after forgetting N , it is a unipotent module in $\text{Mod}_{S_1}^\phi$, we denote this unipotent subcategory by $\text{Mod}_{S_1}^{\phi, N, u}$.

Proposition 2.0.15. *For any $e > 0$, $r < p - 1$, the functor $\text{Mod}_{S_1}^\phi \rightarrow \text{Mod}_{k[u]/u^{ep}}^\phi$ is an equivalence. It transforms short exact sequences to short exact sequences. The functor also induces equivalence on the unipotent subcategories.*

Proof. The equivalence is Proposition 2.3.1 of [Car06]. To check the equivalence on unipotent subcategories, we only need to check that the functor and its inverse sends unipotent objects to unipotent objects, and we can use Remark 1.0.7 for this. To check the exactness, let $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_2 \rightarrow 0$ be a short exact sequence in $\text{Mod}_{S_1}^\phi$, to check short exactness of the resulting sequence, it suffices to check that $\text{Fil}^r \mathcal{M}_1 / \text{Fil}^p S_1 \mathcal{M}_1 \rightarrow \text{Fil}^r \mathcal{M} / \text{Fil}^p S_1 \mathcal{M}$ is injective, i.e., $\text{Fil}^r \mathcal{M}_1 \cap \text{Fil}^p S_1 \mathcal{M} = \text{Fil}^p S_1 \mathcal{M}_1$. This is true because

$$\text{Fil}^p S_1 \mathcal{M}_1 \subseteq \text{Fil}^r \mathcal{M}_1 \cap \text{Fil}^p S_1 \mathcal{M} \subseteq \mathcal{M}_1 \cap \text{Fil}^p S_1 \mathcal{M} = \text{Fil}^p S_1 \mathcal{M}_1.$$

□

Theorem 2.0.16. (1) *When $er = p - 1, r < p - 1$, $\text{Mod}_{S_1}^{\phi, u}$ is an abelian category.*

(2) *When $er = p - 1, r < p - 1$, $\text{Mod}_{S_1}^{\phi, N, u}$ is an abelian category.*

Proof. For statement (1), combine Corollary 1.0.14(2) and Proposition 2.0.15. Statement (2) is a corollary of (1) by keeping track of the N -action. □

For a module $\mathcal{M} \in \underline{\mathcal{M}}^r$, define $\text{Fil}^r p^m \mathcal{M} := p^m \text{Fil}^r \mathcal{M}$, then $(p^m \mathcal{M}, \text{Fil}^r p^m \mathcal{M})$ with the induced ϕ_r and N is an object in $\underline{\mathcal{M}}^r$. For a module $\mathcal{M} \in \underline{\mathcal{M}}^r$, we also define $\mathcal{M}^{(p^m)} := \{x \in \mathcal{M}, p^m x = 0\}$, and let $\text{Fil}^r \mathcal{M}^{(p^m)} := \mathcal{M}^{(p^m)} \cap \text{Fil}^r \mathcal{M}$.

Lemma 2.0.17. *When $r < p - 1$, $\mathcal{M}^{(p)} \in \underline{\mathcal{M}}^r$.*

Proof. The proof follows the same idea as in Lemma 2.3.1.2 of [Bre98], except that we have to tensor with S instead of $W(k)$. We prove the lemma by an induction on the minimal p -power that kills \mathcal{M} . First suppose $p^2 \mathcal{M} = 0$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} S \otimes_{\phi, S} \text{Fil}^r \mathcal{M}^{(p)} & \longrightarrow & S \otimes_{\phi, S} \text{Fil}^r \mathcal{M} & \xrightarrow{\times p} & S \otimes_{\phi, S} p \text{Fil}^r \mathcal{M} & \longrightarrow & 0 \\ \downarrow 1 \otimes \phi_r & & \downarrow 1 \otimes \phi_r & & \downarrow 1 \otimes \phi_r & & \\ 0 \longrightarrow & \mathcal{M}^{(p)} & \longrightarrow & \mathcal{M} & \xrightarrow{\times p} & p \mathcal{M} & \longrightarrow 0 \end{array}$$

Where the top row is right exact and the bottom row is short exact. We only need to show that the vertical arrow on the left is surjective.

Given $x \in \mathcal{M}^{(p)}$, suppose $1 \otimes \phi_r(\hat{x}) = x$ for $\hat{x} \in S \otimes_{\phi, S} \text{Fil}^r \mathcal{M}$, so $1 \otimes \phi_r(p\hat{x}) = 0$. Since $p \mathcal{M} \in \text{Mod}_{S_1}^\phi$, by Lemma 2.2.1 of [Car08] (note that we need $r < p - 1$ here), we must have that $p\hat{x} \in S \otimes_{\phi, S} (E(u)p \text{Fil}^r \mathcal{M} + \text{Fil}^p S \cdot p \mathcal{M})$. So $p\hat{x} = \sum_{i=1}^n a_i \otimes p(E(u)\alpha_i + s_i \beta_i)$, where $a_i \in S, \alpha_i \in \text{Fil}^r \mathcal{M}, s_i \in \text{Fil}^p S, \beta_i \in \mathcal{M}$. Let $\hat{y} = \sum_{i=1}^n a_i \otimes (E(u)\alpha_i + s_i \beta_i)$, then $p\hat{x} = p\hat{y}$, and $1 \otimes \phi_r(\hat{y}) = pz$ for $z \in \mathcal{M}$ (again we need $r < p - 1$ here, so that $\phi_r(\text{Fil}^p S \mathcal{M}) = 0$). Now suppose $1 \otimes \phi_r(\hat{z}) = z$ for $\hat{z} \in S \otimes_{\phi, S} \text{Fil}^r \mathcal{M}$, then $\hat{x} - \hat{y} + p\hat{z} \in S \otimes_{\phi, S} \text{Fil}^r \mathcal{M}^{(p)}$ and maps to x . And we are done.

Now suppose the lemma is true for \mathcal{M} such that $p^m \mathcal{M} = 0$. Then for \mathcal{M} such that $p^{m+1} \mathcal{M} = 0$ and $p^m \mathcal{M} \neq 0$, a similar process as above shows that $\mathcal{M}^{(p^m)} \in \underline{\mathcal{M}}^r$. By induction hypothesis $\mathcal{M}^{(p)} = (\mathcal{M}^{(p^m)})^{(p)} \in \underline{\mathcal{M}}^r$. \square

Now, we define “unipotency” for a module in $\underline{\mathcal{M}}^r$. We define it inductively.

Definition 2.0.18. A module $\mathcal{M} \in \underline{\mathcal{M}}^r$ such that $p^2 \mathcal{M} = 0$ is called unipotent if $\mathcal{M}^{(p)}$ and $p\mathcal{M}$ are unipotent (as modules in $\text{Mod}_{S_1}^\phi$). Inductively, $\mathcal{M} \in \underline{\mathcal{M}}^r$ such that $p^{m+1} \mathcal{M} = 0$ and $p^m \mathcal{M} \neq 0$ is called unipotent if $\mathcal{M}^{(p)}$ and $p\mathcal{M}$ are unipotent. That is, $\mathcal{M}^{(p)}, (p\mathcal{M})^{(p)}, \dots, (p^{m-1}\mathcal{M})^{(p)}$ and $p^m \mathcal{M}$ are all unipotent modules in $\text{Mod}_{S_1}^\phi$. We denote the unipotent subcategory by $\underline{\mathcal{M}}^{r,u}$.

Lemma 2.0.19. Suppose $er = p-1, r < p-1$. For $\mathcal{M} \in \underline{\mathcal{M}}^{r,u}$, we have $p^i \text{Fil}^r \mathcal{M} = p^i \mathcal{M} \cap \text{Fil}^r \mathcal{M}$ for all i .

Proof. We prove by an induction on the minimal p -power that kills \mathcal{M} . Suppose $p^2 \mathcal{M} = 0$ and $p\mathcal{M} \neq 0$. Then there is an injective morphism $p\mathcal{M} \rightarrow \mathcal{M}^{(p)}$. Since $\text{Mod}_{S_1}^{\phi,N,u}$ is abelian, we have $p \text{Fil}^r \mathcal{M} = p\mathcal{M} \cap \text{Fil}^r \mathcal{M}^{(p)} = p\mathcal{M} \cap \text{Fil}^r \mathcal{M} \cap \mathcal{M}^{(p)} = p\mathcal{M} \cap \text{Fil}^r \mathcal{M}$. Suppose the lemma is true for \mathcal{M} such that $p^m \mathcal{M} = 0$. Now suppose $p^{m+1} \mathcal{M} = 0$ and $p^m \mathcal{M} \neq 0$. Then the injective morphism $p^m \mathcal{M} \rightarrow \mathcal{M}^{(p)}$ in the abelian category $\text{Mod}_{S_1}^{\phi,N,u}$ gives us $p^m \text{Fil}^r \mathcal{M} = p^m \mathcal{M} \cap \text{Fil}^r \mathcal{M}^{(p)} = p^m \mathcal{M} \cap \text{Fil}^r \mathcal{M}$. We claim that the module $(\mathcal{M}/p^m \mathcal{M}, \text{Fil}^r \mathcal{M}/p^m \text{Fil}^r \mathcal{M}, \phi_r)$ is in $\underline{\mathcal{M}}^{r,u}$. For any $0 \leq i \leq m-1$, we have the short exact sequence $0 \rightarrow (p^i \mathcal{M})^{(p)}/p^m \mathcal{M} \rightarrow (p^i \mathcal{M}/p^m \mathcal{M})^{(p)} \xrightarrow{h} p^m \mathcal{M} \rightarrow 0$, where for $\bar{x} \in (p^i \mathcal{M}/p^m \mathcal{M})^{(p)}$, take any lift $x \in p^i \mathcal{M}$, and define $h(\bar{x}) = px$. Since $(p^i \mathcal{M})^{(p)}/p^m \mathcal{M}$ and $p^m \mathcal{M}$ are unipotent, $(p^i \mathcal{M}/p^m \mathcal{M})^{(p)}$ is unipotent by Lemma 1.0.12 (and Proposition 2.0.15). And note that $(p^{m-1} \mathcal{M}/p^m \mathcal{M})^{(p)}$ is just $p^{m-1} \mathcal{M}/p^m \mathcal{M}$. So we have proved that $\mathcal{M}/p^m \mathcal{M} \in \underline{\mathcal{M}}^{r,u}$. By the induction hypothesis, $p^i \text{Fil}^r \mathcal{M}/p^m \text{Fil}^r \mathcal{M} = (p^i \mathcal{M}/p^m \mathcal{M}) \cap (\text{Fil}^r \mathcal{M}/p^m \text{Fil}^r \mathcal{M})$ for all i . It is then easy to deduce that $p^i \text{Fil}^r \mathcal{M} = p^i \mathcal{M} \cap \text{Fil}^r \mathcal{M}$ for all i . \square

Theorem 2.0.20. When $er = p-1$ and $r < p-1$, $\underline{\mathcal{M}}^{r,u}$ is an abelian category

The proof of the theorem follows the same strategy as in Section 2.3 of [Bre98]. We have already shown that the subcategory with objects killed by p is abelian, so we only need to do a dévissage argument as in Section 2.3 of [Bre98].

The following lemma will be useful.

Lemma 2.0.21. For $\mathcal{M} \in \underline{\mathcal{M}}^{r,u}$, then $\mathcal{M}/p\mathcal{M} \in \underline{\mathcal{M}}^{r,u}$.

Proof. We prove by an induction on the minimal p -power that kills \mathcal{M} . When $p^2 \mathcal{M} = 0$, then the short exact sequence $0 \rightarrow \mathcal{M}^{(p)}/p\mathcal{M} \rightarrow \mathcal{M}/p\mathcal{M} \xrightarrow{\times p} p\mathcal{M} \rightarrow 0$ concludes the result. Suppose the lemma is true for \mathcal{M} such that $p^m \mathcal{M} = 0$. Now suppose $p^{m+1} \mathcal{M} = 0$ and $p^m \mathcal{M} \neq 0$. Since $p\mathcal{M}$ satisfies the induction hypothesis, $p\mathcal{M}/p^2 \mathcal{M} \in \underline{\mathcal{M}}^{r,u}$. Then the short exact sequence $0 \rightarrow \mathcal{M}^{(p)}/(p\mathcal{M})^{(p)} \rightarrow \mathcal{M}/p\mathcal{M} \xrightarrow{\times p} p\mathcal{M}/p^2 \mathcal{M} \rightarrow 0$ shows that $\mathcal{M}/p\mathcal{M}$ is unipotent. \square

We begin with generalizing Lemma 2.3.1.3 of [Bre98] to the $er = p-1, r < p-1$ unipotent situation.

Lemma 2.0.22. *Let $f : \mathcal{N} \rightarrow \mathcal{M}$ be a morphism in $\underline{\mathcal{M}}^{r,u}$ which is surjective on the S -modules. Suppose that $p\mathcal{M} = 0$. Then $\text{Fil}^r \mathcal{N} \rightarrow \text{Fil}^r \mathcal{M}$ is surjective and $(\mathcal{K}, \text{Fil}^r \mathcal{K}, \phi_r)$ is a kernel of f in $\underline{\mathcal{M}}^{r,u}$, here $\mathcal{K} := \ker f$, $\text{Fil}^r \mathcal{K} := \mathcal{K} \cap \text{Fil}^r \mathcal{N}$.*

Proof. The steps of the proof are exactly the same as Lemma 2.3.1.3 of [Bre98], we just need to check the “unipotency” on each step. For the convenience of the reader, we give a sketch of the proof here. If $p\mathcal{N} = 0$, then the lemma is true because $\text{Mod}_{S_1}^{\phi, N, u}$ is abelian. We do an induction on the minimal p -power that kills \mathcal{N} . Suppose the lemma is true for \mathcal{N} such that $p^{m-1}\mathcal{N} = 0$, and suppose now $p^m\mathcal{N} = 0, p^{m-1}\mathcal{N} \neq 0$. By the same proof as in Lemma 2.3.1.3 of [Bre98], we have $\text{Fil}^r \mathcal{K} \cap p\mathcal{K} = p\text{Fil}^r \mathcal{K}$. The exact sequence $0 \rightarrow \mathcal{K} \cap \mathcal{N}^{(p)} \rightarrow \mathcal{N}^{(p)} \rightarrow \mathcal{M}$ in $\text{Mod}_{S_1}^{\phi, N}$ shows that $\mathcal{K} \cap \mathcal{N}^{(p)} (= \mathcal{K}^{(p)})$ and $\mathcal{N}^{(p)} / \mathcal{K} \cap \mathcal{N}^{(p)}$ are unipotent. The short exact sequence $0 \rightarrow \mathcal{N}^{(p)} / \mathcal{K} \cap \mathcal{N}^{(p)} \rightarrow \mathcal{N} / \mathcal{K} \xrightarrow{\times p} p\mathcal{N} / p\mathcal{K} \rightarrow 0$ shows that $p\mathcal{N} / p\mathcal{K}$ is unipotent. Now $p\mathcal{K} = \ker(p\mathcal{N} \rightarrow p\mathcal{N} / p\mathcal{K})$, since $p^{m-1}(p\mathcal{N}) = 0$, so by the induction hypothesis, $p\mathcal{K} \in \underline{\mathcal{M}}^{r,u}$. Then, the short exact sequence $0 \rightarrow \mathcal{K} / p\mathcal{N} \rightarrow \mathcal{N} / p\mathcal{N} \rightarrow \mathcal{M} \rightarrow 0$ ($\mathcal{N} / p\mathcal{N}$ is unipotent by Lemma 2.0.21) shows that $\mathcal{K} / p\mathcal{N}$ is unipotent. And the short exact sequence $0 \rightarrow p\mathcal{N} / p\mathcal{K} \rightarrow \mathcal{K} / p\mathcal{K} \rightarrow \mathcal{K} / p\mathcal{N} \rightarrow 0$ of S_1 -modules shows that $\mathcal{K} / p\mathcal{K}$ is finite free over S_1 . By Lemma 2.3.1.1 of [Bre98], $\mathcal{K} \in \underline{\mathcal{M}}^r$. And we are done. \square

Proof of Theorem 2.0.20. Firstly we claim that if $f : \mathcal{N} \rightarrow \mathcal{M}$ is a morphism in $\underline{\mathcal{M}}^{r,u}$, then $(\mathcal{K}, \text{Fil}^r \mathcal{K}, \phi_r)$ is a kernel of f in $\underline{\mathcal{M}}^{r,u}$. But this is just the $er = p - 1, r < p - 1$ unipotent generalization of Proposition 2.3.2.1 of [Bre98]. Just apply lemma 2.0.22, and the proof is almost verbatim.

Secondly we claim that if $f : \mathcal{N} \rightarrow \mathcal{M}$ is a morphism in $\underline{\mathcal{M}}^{r,u}$, then $f(\text{Fil}^r \mathcal{N}) = \text{Fil}^r \mathcal{M} \cap f(\mathcal{N})$, and the naturally defined cokernel $(\mathcal{C}, \text{Fil}^r \mathcal{C}, \phi_r)$ is a cokernel of f in $\underline{\mathcal{M}}^{r,u}$. And this is a generalization of Proposition 2.3.2.2 of [Bre98] to the $er = p - 1, r < p - 1$ unipotent situation. The reader can easily check that everything works through if we check “unipotency” at each step. \square

Proposition 2.0.23. *Suppose $er = p - 1$ and $r < p - 1$. Let $0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_2 \rightarrow 0$ be a short exact sequence in $\underline{\mathcal{M}}^r$ where $\mathcal{M}_1, \mathcal{M}_2$ are unipotent. Then \mathcal{M} is also unipotent.*

Proof. Apply snake lemma to the following commutative diagram,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}_2 \longrightarrow 0 \\ & & \downarrow \times p & & \downarrow \times p & & \downarrow \times p \\ 0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}_2 \longrightarrow 0. \end{array}$$

We have the exact sequence $0 \rightarrow \mathcal{M}_1^{(p)} \rightarrow \mathcal{M}^{(p)} \rightarrow \mathcal{M}_2^{(p)} \xrightarrow{h} \mathcal{M}_1 / p\mathcal{M}_1$, where h is the connecting homomorphism. Then it is easy to conclude that $\mathcal{M}^{(p)} \in \underline{\mathcal{M}}^{r,u}$. We now prove that $p\mathcal{M}$ is unipotent. We prove this by an induction on the minimal p -power that kills \mathcal{M}_1 . If $p\mathcal{M}_1 = 0$, then by the short exact sequence $0 \rightarrow \mathcal{M}^{(p)} / \mathcal{M}_1 \rightarrow \mathcal{M} / \mathcal{M}_1 (= \mathcal{M}_2) \rightarrow \mathcal{M} / \mathcal{M}^{(p)} \rightarrow 0$, and since $\underline{\mathcal{M}}^{r,u}$ is abelian, $p\mathcal{M} = \mathcal{M} / \mathcal{M}^{(p)} \in \underline{\mathcal{M}}^{r,u}$. Suppose the proposition is true for \mathcal{M}_1 such that $p^m\mathcal{M}_1 = 0$. Now suppose $p^{m+1}\mathcal{M}_1 = 0$ and $p^m\mathcal{M}_1 \neq 0$. The short exact sequence $0 \rightarrow \mathcal{M}^{(p)} / \mathcal{M}_1^{(p)} \rightarrow \mathcal{M} / \mathcal{M}_1 \xrightarrow{\times p} p\mathcal{M} / p\mathcal{M}_1 \rightarrow 0$ shows that $p\mathcal{M} / p\mathcal{M}_1 \in \underline{\mathcal{M}}^{r,u}$.

Then apply induction hypothesis to the short exact sequence $0 \rightarrow p\mathcal{M}_1 \rightarrow p\mathcal{M} \rightarrow p\mathcal{M}/p\mathcal{M}_1 \rightarrow 0$ to conclude that $p\mathcal{M}$ is unipotent. \square

Corollary 2.0.24. *Let \mathcal{C} be the smallest full subcategory of $\underline{\mathcal{M}}^r$ which contains $\text{Mod}_{S_1}^{\phi, N, u}$ and is stable by extension. Then \mathcal{C} is the same as $\underline{\mathcal{M}}^{r, u}$.*

Proof. By Proposition 2.0.23, \mathcal{C} is a subcategory of $\underline{\mathcal{M}}^{r, u}$. Now for any object $\mathcal{M} \in \underline{\mathcal{M}}^{r, u}$, by an easy induction process on the minimal p -power that kills \mathcal{M} , and using the short exact sequence $0 \rightarrow p\mathcal{M} \rightarrow \mathcal{M} \rightarrow \mathcal{M}/p\mathcal{M} \rightarrow 0$ (i.e., \mathcal{M} is an extension of $p\mathcal{M}$ and $\mathcal{M}/p\mathcal{M}$), it is clear that $\mathcal{M} \in \mathcal{C}$. \square

3. A REMARK ON TORSION LOG-SYNTOMIC COHOMOLOGIES

In this section, we give a brief sketch of some of the work on torsion log-syntomic cohomologies in [Car08], and point out why the approach in *loc. cit.* cannot be generalized to $er = p - 1$.

Let X be a proper semi-stable scheme over \mathcal{O}_K . In [Car08], Caruso proves a “comparison” theorem (Theorem 1.1 of [Car08]) between the torsion étale cohomology of $X_{\overline{K}}$ and certain log-crystalline cohomology of X . The key ingredient (Theorem 4.2.1 in [Car08]) is to show certain torsion log-syntomic cohomology of X gives rise to torsion Breuil modules (modules in the category $\underline{\mathcal{M}}^r$), under the condition $er < p - 1$. Here, we give a very rough sketch of the proof, and point out why the approach can not be generalized to $er = p - 1$.

Here, we only introduce notations from [Car08] that is necessary for our purpose. The reader should consult [Car08] for more details.

Suppose B is a log scheme such that the underlying scheme is killed by some nonzero integer, and it is equipped with an ideal of divided powers. Suppose X is a finite log scheme over B such that the divided powers extend. We use $(X/B)_{\text{cris}}$ to denote the small finite log-crystalline site. Let $\mathcal{O}_{X/B}$ be the structure sheaf over the site, $\mathcal{I}_{X/B}$ the ideal of divided powers, and $\mathcal{I}_{X/B}^{[n]}$ the successive divided powers of $\mathcal{I}_{X/B}$. If B is a finite log scheme, we use B_{SYN} (resp. B_{syn}) to denote the big (resp. small) log-syntomic site over B .

We use $W_n, T, T_n, E_n, \tilde{E}$ and $\tilde{E}_{(2)}$ to denote the following log schemes:

$$(0 \rightarrow W_n), \left(\begin{array}{c} \mathbb{N} \rightarrow \mathcal{O}_K \\ 1 \mapsto \pi \end{array} \right), \left(\begin{array}{c} \mathbb{N} \rightarrow \mathcal{O}_K/p^n \\ 1 \mapsto \pi \end{array} \right), \left(\begin{array}{c} \mathbb{N}u \rightarrow S_n \\ u \mapsto u \end{array} \right), \left(\begin{array}{c} \mathbb{N}u \rightarrow \tilde{S} \\ u \mapsto u \end{array} \right), \left(\begin{array}{c} \mathbb{N}u \rightarrow \tilde{S}_{(2)} \\ u \mapsto u \end{array} \right).$$

Here $S_n = S/p^n S$, $\tilde{S} = k[u]/u^p$, $\tilde{S}_{(2)} = k[u]/u^{2p}$.

Now, suppose X is a proper semi-stable scheme over \mathcal{O}_K , then its special fiber defines a log structure. X is a log-smooth log scheme over T . For any n , let $X_n = X \times_T T_n$. We define sheaves $\mathcal{I}_n^{[s]}$ over the site $(T_n)_{\text{SYN}}$ by the association $U \mapsto H^0((U/E_n)_{\text{cris}}, \mathcal{I}_{U/E_n}^{[s]})$, and denote $\mathcal{O}_n^{\text{st}} = \mathcal{I}_n^{[0]}$.

Theorem 3.0.25 ([Car08]). *When $er < p - 1$, $i < r$, then*

$$\mathcal{M}_n^{(i)} = (H^i((X_n)_{\text{syn}}, \mathcal{O}_n^{\text{st}}), H^i((X_n)_{\text{syn}}, \mathcal{I}_n^{[r]}), \phi_r, N)$$

defines an object in $\underline{\mathcal{M}}^r$, where ϕ_r and N are the operators as defined in Section 3.4 of [Car08].

To prove the theorem, one starts from $n = 1$, and proves that $\mathcal{M}_1^{(i)} \in \underline{\mathcal{M}}^r$ for $i \leq r$ (Theorem 4.1.24 of [Car08]). By using that $\underline{\mathcal{M}}^r$ is an abelian category when $er < p - 1$, a dévissage argument as in Section 4.2 of [Car08] will finish the proof.

In order to prove that $\mathcal{M}_1^{(i)} \in \underline{\mathcal{M}}^r$, one utilizes the other two bases \tilde{E} and $\tilde{E}_{(2)}$. Let Υ be either \tilde{E} and $\tilde{E}_{(2)}$, and define sheaves $\mathcal{J}_\Upsilon^{[s]}$ over the site $(T_1)_{\text{SYN}}$ by the association $U \mapsto H^0((U/\Upsilon)_{\text{cris}}, \mathcal{J}_{U/\Upsilon}^{[s]})$, and denote $\mathcal{O}_\Upsilon^{\text{st}} = \mathcal{J}_\Upsilon^{[0]}$.

Let

$$\mathcal{M}_\Upsilon^{(i)} := (H^i((X_1)_{\text{SYN}}, \mathcal{O}_\Upsilon^{\text{st}}), H^i((X_1)_{\text{SYN}}, \mathcal{J}_\Upsilon^{[r]}), \phi_r, N).$$

Then Theorem 4.1.17 of [Car08] proves that for $i \leq r$, $\mathcal{M}_{\tilde{E}}^{(i)}$ is an object in $\text{Mod}_{k[u]/u^p}^\phi$ (see our Definition 1.0.1) with additional natural N -structure. In fact, we can easily check that all the steps (from Proposition 4.1.11 to Proposition 4.1.16 in [Car08]) leading to the proof of Theorem 4.1.17 of *loc. cit.* still work for $er = p - 1$! So we have,

Theorem 3.0.26 ([Car08]). *When $er \leq p - 1, r < p - 1$ and $i \leq r$, $\mathcal{M}_{\tilde{E}}^{(i)}$ is an object in $\text{Mod}_{k[u]/u^p}^\phi$ with additional natural N -structure.*

The next step (Theorem 4.1.22 in [Car08]) is to prove that for $i \leq r$, $\mathcal{M}_{\tilde{E}_{(2)}}^{(i)}$ is an object in $\text{Mod}_{k[u]/u^{2p}}^\phi$ (see our Definition 1.0.1) with additional natural N -structure. It is reduced (by Theorem 4.1.1 of [Car08]) to calculating log-syntomic cohomology over the base \tilde{E} by the fact $\mathcal{J}_{\tilde{E}_{(2)}}^{[r]}/\mathcal{J}_{\tilde{E}_{(2)}}^{[r+2/e]} \simeq \mathcal{J}_{\tilde{E}}^{[r]}/\mathcal{J}_{\tilde{E}}^{[r+2/e]}$ (Corollary 3.3.5 of [Car08], and see Definition 3.3.3 of *loc. cit.* for these notations). This is where we must have $er < p - 1$.

To see the failure from a simple fact, recall that $T_s = k[u]/u^s$. Then for $T_{2p}(= \tilde{S}_{(2)})$ and $T_p(= \tilde{S})$, we have $\text{Fil}^a T_{2p}/\text{Fil}^b T_{2p}$ and $\text{Fil}^a T_p/\text{Fil}^b T_p$ are isomorphic if $eb \leq p$, but **not** so when $eb > p$! And in our situation $e(r + 2/e) = er + 2$.

Remark 3.0.27. Note that we cannot say Theorem 3.0.25 is wrong when $er = p - 1$. We only pointed out that the approach in [Car08] breaks when $er = p - 1$. However, as we pointed out in Theorem 3.0.26, some of the work in [Car08] readily goes through when $er = p - 1$. So we hope that writing down this section can still be useful for later research. It is possible that we can prove Theorem 3.0.25 without using the base $\tilde{E}_{(2)}$, and so avoiding the above failure. We hope to come back to this in later work.

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